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Modelling streamline curvature effects in explicit algebraic Reynolds stress turbulence models

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Abstract

A curvature correction for explicit algebraic Reynolds stress models (EARSMs), based on a formal derivation of the weak-equilibrium assumption in a streamline oriented curvilinear co-ordinate system is presented. The curvature correction is given from the rotation rate of the curvilinear co-ordinate system following the mean flow. Two methods for defining that rotation rate are proposed, one is derived from the strain-rate tensor, and the other is derived from the local mean acceleration vector. Both methods are fully three-dimensional and Galilean invariant and the correction vanishes in cases without curvature or rotation effects. The EARSM proposed by Wallin and Johansson (J. Fluid Mech. 403 (2000) 89) was extended with the proposed curvature corrections and recalibrated in such a way that the original model was retrieved in cases without curvature or rotation effects. Rotating homogeneous turbulent shear flows with vanishing mean vorticity should be close to neutral stability according to linear stability theory, also observed from large eddy simulations. This was used for the recalibration. The importance of the curvature correction and the proposed recalibration is shown for rotating homogeneous shear and rotating channel flows.

1. Introduction

Turbulent flows over curved surfaces, near stagnation and separation points, in vortices and turbulent flows in rotating frames of reference are all affected by streamline curvature effects. Strong curvature and/or rotational effects form a major cornerstone problem also at the Reynolds stress transport modelling level, and pressure–strain rate models that are able to accurately capture rapidly rotating turbulence are rare. In more moderate situations the SSG model (Speziale et al., 1991), and derivations thereof, show rather good behaviour in rotating flows such as rotating homogeneous shear flows (see Gatski and Speziale, 1993). Standard eddy-viscosity models without explicit corrections, completely fail in describing effects of local as well as global rotation.

Algebraic Reynolds stress models (Rodi, 1972, 1976) are the results of applying the weak-equilibrium assumption on the full differential models. In the weak-equilibrium limit of turbulence, the Reynolds stress anisotropy tensor, $a_{ij} \equiv \overline{u_i u_j}/K - (2/3)\delta_{ij}$, is assumed to be constant following a streamline. Neglecting also the diffusion of the anisotropy tensor results in an implicit purely algebraic relation for a_{ij}. Algebraic modelling has had a renewal during the last decade after it was found that the resulting implicit algebraic relation for a_{ij} may be formally solved resulting in an explicit relation (see e.g. Pope, 1975; Taulbee, 1992; Gatski and Speziale, 1993; Girimaji, 1996; Johansson and Wallin, 1996).

The material derivative that includes advection by the mean flow (in the following denoted $\mathrm{D}/\mathrm{D}t$) of a scalar field is invariant of the choice of co-ordinate system. However, the material derivative of a tensor field of higher rank than zero, e.g. vectors and second rank tensors, is *not* invariant of the choice of co-ordinate system for representing the tensor components.

It has been suggested by e.g. Rodi and Scheurer (1983) that the weak-equilibrium assumption is better evaluated for the anisotropy tensor expressed in a streamline-based co-ordinate system. In e.g. circular flows where the azimuthal direction is homogeneous, the weak-equilibrium assumption is then exactly fulfilled. This approach is, however, not Galilean invariant and should not be used in a general model.

Galilean invariant methods have been proposed by Girimaji (1997) and Gatski and Jongen (2000) which are

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Nomenclature

a_{ij} Reynolds stress anisotropy tensor, $\overline{u_i u_j}/K - (2/3)\delta_{ij}$
$\mathscr{D}_{ij}^{(a)}$ diffusion of a_{ij}
$\mathrm{D}/\mathrm{D}t$, $(\dot{})$ advection by mean flow, $\partial/\partial t + U_j \partial/\partial x_j$
e_{ij} dissipation rate anisotropy, $\varepsilon_{ij}/\varepsilon - (2/3)\delta_{ij}$
$\boldsymbol{e}_{\mathrm{s}}$ curvilinear co-ordinate system, $(\hat{\boldsymbol{t}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{s}})$
K turbulent kinetic energy, $\overline{u_i u_i}/2$
$\mathscr{P}$ turbulence production
Ro rotation number
Ro^-, Ro^+ bifurcation rotation numbers
Re_τ Reynolds number based on friction velocity
Re_m Reynolds number based on bulk velocity
S_{ij} normalized mean flow strain-rate tensor, $\tau(U_{i,j} + U_{j,i})/2$
$\boldsymbol{T}$ orthogonal transformation
$\overline{u_i u_j}$ Reynolds stress tensor
u_τ wall friction velocity
ε dissipation rate of K
Γ vortex circulation, $2\pi r V$
$\boldsymbol{\Lambda}$ eigenvalue tensor (diagonal)
Π_{ij} pressure–strain rate
τ turbulent time scale, K/ε
Ω_{ij} normalized mean flow rotation rate tensor, $\tau(U_{i,j} - U_{j,i})/2$
$\boldsymbol{\Omega}^{(\mathrm{r})}$ co-ordinate system rotation rate tensor
$\boldsymbol{\Omega}^{*}$ effective mean flow rotation rate tensor
$\boldsymbol{\omega}^{(\mathrm{r})}$ co-ordinate system rotation rate vector

Subscript
s in the curvilinear co-ordinate system

Superscripts
t transpose
$+$ normalized by wall units

based on the rotation rate of the acceleration vector and the strain-rate tensor, respectively, following the mean flow. The latter is an extension of the ideas by Spalart and Shur (1997) for correcting eddy-viscosity models. These methods are, in principal, derived for general three-dimensional (3D) flows, but so far, only complete expressions and methods for two-dimensional (2D) mean flows have been presented. These ideas are extended in the following section and fully 3D closed form relations are presented.

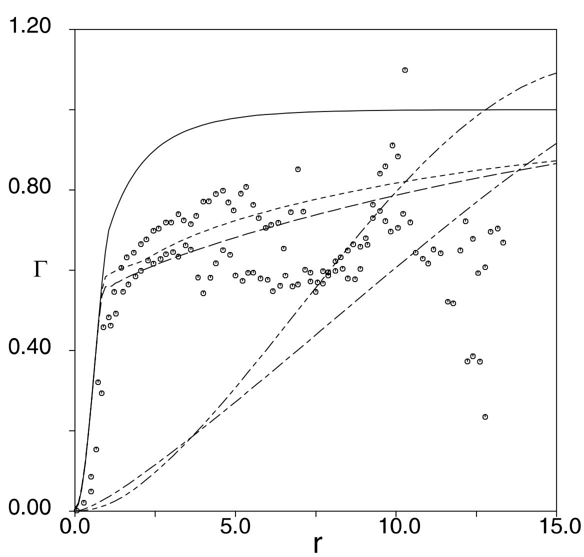

Fig. 1. Computed vortex circulation $\Gamma \equiv 2\pi r V$ compared to field measurements (Campbell et al., 1996, 1997) with the initial profile (—). RST (– – –) compared to the corresponding EARSM with (- - -) and without (– - – - –) streamline correction and standard K–ε model (– - - - - - –). Taken from Wallin and Girimaji (2000).

In cases with moderately curved streamlines the choice of co-ordinate system has a rather minor effect, see e.g. Rumsey et al. (1999) for flow over an airfoil. However, in cases with strong streamline curvature this effect is dominating. In a study of a generic wing tip far-field vortex by Wallin and Girimaji (2000) it was found that the turbulent dissipation of the vortex was by far overpredicted using the standard algebraic Reynolds stress models while including the effect of the streamline curvature gave a qualitatively correct behaviour (see Fig. 1).

2. Curvature corrected model

General quasi-linear Reynolds stress transport models may be written in terms of a transport equation for the anisotropy tensor

$$\tau\left(\frac{\mathrm{D}a_{ij}}{\mathrm{D}t} - \mathscr{D}_{ij}^{(a)}\right) = A_0\left[\left(A_3 + A_4\frac{\mathscr{P}}{\varepsilon}\right)a_{ij} + A_1 S_{ij} - \left(a_{ik}\Omega_{kj} - \Omega_{ik}a_{kj}\right) + A_2\left(a_{ik}S_{kj} + S_{ik}a_{kj} - \frac{2}{3}a_{kl}S_{lk}\delta_{ij}\right)\right] \qquad (1)$$

(see Wallin and Johansson, 2000). $\mathscr{D}_{ij}^{(a)}$ is the diffusion of a_{ij} and $\tau = K/\varepsilon$ is the turbulent time scale. The strain and rotation rate tensors, S_{ij} and Ω_{ij}, are normalized by τ. This relation results from the general quasi-linear model for the pressure–strain rate and dissipation rate anisotropy, e_{ij}, lumped together

$$\frac{\Pi_{ij}}{\varepsilon} - e_{ij} = -\frac{1}{2}\left(C_1^0 + C_1^1\frac{\mathscr{P}}{\varepsilon}\right)a_{ij} + C_2 S_{ij} + \frac{C_3}{2}\left(a_{ik}S_{kj} + S_{ik}a_{kj} - \frac{2}{3}a_{kl}S_{lk}\delta_{ij}\right) - \frac{C_4}{2}\left(a_{ik}\Omega_{kj} - \Omega_{ik}a_{kj}\right). \quad (2)$$

The A coefficients are related to the C coefficients through

$$A_0 = \frac{C_4}{2} - 1, \quad A_1 = \frac{3C_2 - 4}{3A_0}, \quad A_2 = \frac{C_3 - 2}{2A_0}, \quad A_3 = \frac{2 - C_1^0}{2A_0}, \quad A_4 = \frac{-C_1^1 - 2}{2A_0}. \quad (3)$$

2.1. The weak-equilibrium assumption

Usually, in deriving algebraic Reynolds stress models the *l.h.s.* of (1) is neglected in the computational (=Cartesian) co-ordinate system. The resulting algebraic relation may be formally solved leading to an explicit algebraic Reynolds stress model (EARSM), that is an explicit relation for a_{ij} (see Wallin and Johansson, 2000). The A_0 coefficient in (1) influences the EARSM only if a contribution of the l.h.s. of (1) is included into the EARSM.

Girimaji (1997) and Sjögren (1997) realized that, by imposing the weak-equilibrium assumption in a general curvilinear co-ordinate system, $\boldsymbol{e}_s \equiv (\hat{\boldsymbol{t}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{s}})$, an additional algebraic term appears which easily can be accounted for in the EARSM solution.

The Cartesian (or computational) co-ordinate system $\boldsymbol{e} \equiv (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ is transformed to $\boldsymbol{e}_s = \boldsymbol{T}\boldsymbol{e}$ using the orthogonal transformation $\boldsymbol{T}$ ($\boldsymbol{T}^t\boldsymbol{T} = \boldsymbol{I}$). The advection of the anisotropy (or any second rank tensor) may be expanded to

$$\frac{\mathrm{D}\boldsymbol{a}}{\mathrm{D}t} = \boldsymbol{T}^t \frac{\mathrm{D}\boldsymbol{T}\boldsymbol{a}\boldsymbol{T}^t}{\mathrm{D}t}\boldsymbol{T} - \left(\boldsymbol{a}\boldsymbol{\Omega}^{(r)} - \boldsymbol{\Omega}^{(r)}\boldsymbol{a}\right), \quad (4)$$

where the first term on the *r.h.s.* is interpreted as the advection of the transformed anisotropy, $\boldsymbol{a}_s = \boldsymbol{T}\boldsymbol{a}\boldsymbol{T}^t$, transformed back to the Cartesian system, $\boldsymbol{T}^t(\cdots)\boldsymbol{T}$. The second term contains the antisymmetric tensor

$$\boldsymbol{\Omega}^{(r)} = \frac{\mathrm{D}\boldsymbol{T}^t}{\mathrm{D}t}\boldsymbol{T} = -\boldsymbol{T}^t\frac{\mathrm{D}\boldsymbol{T}}{\mathrm{D}t}. \quad (5)$$

If the advection of the anisotropy is neglected in the $\boldsymbol{e}_s$ system, the resulting term $(\boldsymbol{a}\boldsymbol{\Omega}^{(r)} - \boldsymbol{\Omega}^{(r)}\boldsymbol{a})$ may be fully accounted for and included into the EARSM formulation simply by replacing $\boldsymbol{\Omega}$ in (1) with

$$\boldsymbol{\Omega}^* = \boldsymbol{\Omega} - \frac{\tau}{A_0}\boldsymbol{\Omega}^{(r)}. \quad (6)$$

The formal transformation of the material derivative has earlier been presented by Girimaji (1997) and Sjögren (1997), the later with an extension for non-orthogonal co-ordinates. They extended this analysis with a formal derivation of the $\boldsymbol{\Omega}^{(r)}$ tensor from the definition of the co-ordinate system. That analysis becomes rather tedious especially in general 3D flows. The final expression includes derivatives of the metrics of the curvilinear co-ordinate system which may cause numerical problems when utilized for real 3D numerical computations (personal communications T. Rung, Technical University of Berlin).

In this paper we will not go further into the derivation of $\boldsymbol{\Omega}^{(r)}$, but rather try to understand the physical interpretation of $\boldsymbol{\Omega}^{(r)}$. The material derivative of $\boldsymbol{e}_s$ is expressed by using the transformation $\boldsymbol{T}$

$$\frac{\mathrm{D}\boldsymbol{e}_s}{\mathrm{D}t} = \frac{\mathrm{D}\boldsymbol{T}}{\mathrm{D}t}\boldsymbol{T}^t\boldsymbol{e}_s = -\boldsymbol{\Omega}_s^{(r)}\boldsymbol{e}_s, \quad (7)$$

where $\boldsymbol{\Omega}_s^{(r)} = \boldsymbol{T}\boldsymbol{\Omega}^{(r)}\boldsymbol{T}^t$ is $\boldsymbol{\Omega}^{(r)}$ transformed to the curvilinear co-ordinate system. The $\boldsymbol{\Omega}^{(r)}$ tensor is, thus, directly related to the rotation rate of the co-ordinate system following the fluid particle

$$\Omega_{ij}^{(r)} = -\epsilon_{ijk}\omega_k^{(r)}, \quad (8)$$

where $\boldsymbol{\omega}^{(r)}$ is the co-ordinate system rotation rate vector, also noted by Gatski and Jongen (2000).

The problem is now reduced to finding the $\boldsymbol{\omega}^{(r)}$ that minimizes the advection of the anisotropy tensor in the $\boldsymbol{e}_s$ system. In general 3D flow fields it is not possible to know the optimal transformation a priori, and, thus, $\boldsymbol{\omega}^{(r)}$ must be derived from the computed flow field.

2.2. Strain-rate based co-ordinate system

If we assume that there is an algebraic relation for $\boldsymbol{a}$ in terms of $\boldsymbol{S}$ and $\boldsymbol{\Omega}^*$, $\boldsymbol{a} = f(\boldsymbol{S}, \boldsymbol{\Omega}^*)$, the variation of $\boldsymbol{a}$ might be expressed in terms of the variation of $\boldsymbol{S}$ and $\boldsymbol{\Omega}^*$. If there exist a curvilinear co-ordinate system where the variation of $\boldsymbol{S}$ and $\boldsymbol{\Omega}^*$ vanishes in the mean flow direction, then also the variation of $\boldsymbol{a}$ must vanish in that system. Let us try to find the curvilinear co-ordinate system, or equivalent the $\boldsymbol{\Omega}^{(r)}$ tensor, for which the variation of $\boldsymbol{S}$ is minimal. The question of considering also the variation of $\boldsymbol{\Omega}^*$ will be discussed later.

The advection of the strain-rate tensor $\boldsymbol{S}$ may, similarly to the advection of the anisotropy tensor (4), be expressed as the advection in a curvilinear co-ordinate system plus an algebraic term arising from the transformation

$$\frac{\mathrm{D}\boldsymbol{S}}{\mathrm{D}t} = \boldsymbol{T}^t\frac{\mathrm{D}\boldsymbol{T}\boldsymbol{S}\boldsymbol{T}^t}{\mathrm{D}t}\boldsymbol{T} - \left(\boldsymbol{S}\boldsymbol{\Omega}^{(r)} - \boldsymbol{\Omega}^{(r)}\boldsymbol{S}\right). \quad (9)$$

The best approximation of the co-ordinate system where the advection of the $\boldsymbol{S}$ tensor is neglected may be obtained by finding the solution for the $\boldsymbol{\Omega}^{(r)}$ tensor from (9) where the first term on the r.h.s. is set to zero. However, that equation system is overdetermined since there are five (two in 2D) independent equations for

$\dot{\boldsymbol{S}} \equiv \mathrm{D}\boldsymbol{S}/\mathrm{D}t$ and three (one in 2D) independent components of $\boldsymbol{\Omega}^{(\mathrm{r})}$.

Let $\dot{S}'_{ij}$ be the advection of the transformed S_{ij} (first term on the r.h.s. of (9)). By using that $\boldsymbol{\Omega}^{(\mathrm{r})} \equiv -\epsilon_{ijk}\omega_k^{(\mathrm{r})}$ the equation for $\dot{S}'_{ij}$ becomes

$$\dot{S}'_{ij} = \dot{S}_{ij} - (S_{il}\epsilon_{ljk} + S_{jl}\epsilon_{lik})\omega_k^{(\mathrm{r})}. \tag{10}$$

$\dot{S}'_{ij}$ may be minimized in a least square sense by minimizing the norm $\dot{S}'_{ij}\dot{S}'_{ij}$, which, for this case, is equivalent with $S_{pl}\dot{S}'_{lq}\epsilon_{pqi} = 0$. That results in

$$S_{pl}\dot{S}_{lq}\epsilon_{pqi} = A_{ij}\omega_j^{(\mathrm{r})}, \tag{11}$$

where $A_{ij} = 2II_S\delta_{ij} - 3S_{ik}S_{kj}$ (derived by using the relation $\epsilon_{ijk}\epsilon_{rst} = \delta_{ir}\delta_{js}\delta_{kt} + \delta_{is}\delta_{jt}\delta_{kr} + \delta_{it}\delta_{jr}\delta_{ks} - \delta_{ir}\delta_{jt}\delta_{ks} - \delta_{js}\delta_{it}\delta_{kr} - \delta_{kt}\delta_{is}\delta_{jr}$).

The solution of (10) for the rotation vector $\omega_i^{(\mathrm{r})}$ can now be determined by multiplying by the inverse of A_{ij}

$$\omega_i^{(S)} = A_{ij}^{-1}S_{pl}\dot{S}_{lq}\epsilon_{pqj}, \tag{12}$$

where A_{ij} is inverted by the aid of the Caley–Hamilton theorem to

$$A_{ij}^{-1} = \frac{II_S^2\delta_{ij} + 12III_S S_{ij} + 6II_S S_{ik}S_{kj}}{2II_S^3 - 12III_S^2}. \tag{13}$$

An interesting observation is that if the transformation $\boldsymbol{T}$ is the eigenvectors of $\boldsymbol{S}$, then the transformed $\boldsymbol{S}$ is the diagonal eigenvalue tensor $\boldsymbol{\Lambda}$. The first term on the r.h.s. in (9) then becomes $\boldsymbol{T}^{\mathrm{t}}\dot{\boldsymbol{\Lambda}}\boldsymbol{T}$. Multiplying that term by $\boldsymbol{S}$ results in $\boldsymbol{S}\boldsymbol{T}^{\mathrm{t}}\dot{\boldsymbol{\Lambda}}\boldsymbol{T} = \boldsymbol{T}^{\mathrm{t}}\boldsymbol{\Lambda}\dot{\boldsymbol{\Lambda}}\boldsymbol{T}$, which is symmetric since both $\boldsymbol{\Lambda}$ and $\dot{\boldsymbol{\Lambda}}$ are real. That term, thus, vanishes when multiplied by ϵ_{ijk}. In other words, the transformation $\boldsymbol{T}$ that minimizes $\dot{S}'_{ij}\dot{S}'_{ij}$ is given by the eigenvectors of $\boldsymbol{S}$.

The proposed method is, thus, identical to the Gatski and Jongen (2000) assumption of relating $\boldsymbol{\omega}^{(\mathrm{r})}$ to the rotation rate of the principal directions of $\boldsymbol{S}$ following the mean flow. While Gatski and Jongen presented an explicit relation for $\boldsymbol{\omega}^{(\mathrm{r})}$ only for 2D mean flows, the present relation (12) is valid also for 3D flows.

The denominator in (13) may be investigated for the case where $\boldsymbol{S}$ is oriented in the principal direction. $\boldsymbol{S}$ is then diagonal with the components $[\alpha, \beta, -\alpha-\beta]$. The denominator then becomes

$$2II_S^3 - 12III_S^2 = 4(\alpha-\beta)^2(2\alpha+\beta)^2(\alpha+2\beta)^2 \tag{14}$$

which is positive for all α and β except when two of the eigenvalues are equal or all eigenvalues are zero. The singularities at these points may be avoided by adding a small number to the denominator.

In 2D mean flows, $\omega_i^{(S)}$ reduces to

$$\omega_3^{(S)} = \frac{S_{11}\dot{S}_{12} - S_{12}\dot{S}_{11}}{2S_{11}^2 + 2S_{12}^2} \tag{15}$$

which is identical to the Spalart and Shur (1997) and the Gatski and Jongen (2000) corrections.

The variation of $\boldsymbol{\Omega}^*$ may also be considered in determining the optimal $\omega_i^{(\mathrm{r})}$. By minimizing the norm $\dot{S}'_{ij}\dot{S}'_{ij} + \dot{\Omega}'*_{ij}\dot{\Omega}'*_{ij}$, where $\dot{\Omega}'*_{ij}$ is the advection of the transformed Ω_{ij}, the transformation $\omega^{(S-\Omega)}$ that minimizes both the variation of $\boldsymbol{S}$ and $\boldsymbol{\Omega}$ is found. In 2D mean flows, $\boldsymbol{\omega}^{(S-\Omega)}$ reduces to $\boldsymbol{\omega}^{(S)}$ in (15), but in general $\boldsymbol{\omega}^{(S-\Omega)} \neq \boldsymbol{\omega}^{(S)}$. However, due to the huge algebraic complexity, the complete $\boldsymbol{\omega}^{(S-\Omega)}$ is of limited practical use.

2.3. Acceleration based system

It was proposed by Girimaji (1997) to use the acceleration vector $\dot{\boldsymbol{U}} \equiv \mathrm{D}\boldsymbol{U}/\mathrm{D}t$ as the basis for constructing the curvilinear co-ordinate system. Since the acceleration vector is Galilean invariant the resulting streamline curvature corrected model would also be Galilean invariant. Girimaji further suggested to let one of the other unit vectors be in the direction of $\dot{\boldsymbol{U}} \times \ddot{\boldsymbol{U}}$. In 2D mean flows that direction is known, but in 3D flows, one needs to derive $\ddot{\boldsymbol{U}} \equiv \mathrm{D}\dot{\boldsymbol{U}}/\mathrm{D}t$ from the mean flow field. Since $\boldsymbol{\Omega}^{(\mathrm{r})}$ also depend on the rate of change of the unit vectors, one additional derivative of the velocity field is needed in the resulting, quite complex, expression.

Is it possible to approximate the co-ordinate system rotation rate, $\boldsymbol{\omega}^{(\mathrm{r})}$, directly from the acceleration vector and the rate of change of that? Let us investigate the following approximation proposed by Wallin (2000)

$$\boldsymbol{\omega}^{(\mathrm{approx})} = \frac{\dot{\boldsymbol{U}} \times \ddot{\boldsymbol{U}}}{\dot{\boldsymbol{U}}^2}. \tag{16}$$

This approximation obviously gives the correct $\boldsymbol{\omega}^{(\mathrm{r})}$ in circular flows and was the motivation behind the approximation.

The approximation may be related to the acceleration based system, $\boldsymbol{\omega}^{(\mathrm{acc})}$, proposed by Girimaji. In the curvilinear co-ordinate system, let $\hat{\boldsymbol{n}}$ be in the direction of the acceleration $\dot{\boldsymbol{U}}$, $\hat{\boldsymbol{s}}$ in the direction of $\dot{\boldsymbol{U}} \times \ddot{\boldsymbol{U}}$ and $\hat{\boldsymbol{t}} = \hat{\boldsymbol{n}} \times \hat{\boldsymbol{s}}$. The acceleration vector may then be written as $\dot{\boldsymbol{U}} = a\hat{\boldsymbol{n}}$, where $a = |\dot{\boldsymbol{U}}|$. The rate of change of the acceleration is $\ddot{\boldsymbol{U}} = \dot{a}\hat{\boldsymbol{n}} + a\dot{\hat{\boldsymbol{n}}}$ and the approximation then becomes

$$\boldsymbol{\omega}^{(\mathrm{approx})} = \hat{\boldsymbol{n}} \times \dot{\hat{\boldsymbol{n}}} = \boldsymbol{\omega}^{(\mathrm{acc})} - \omega_n^{(\mathrm{acc})}\hat{\boldsymbol{n}}, \tag{17}$$

where we have used relations (7) and (8) for expressing $\dot{\hat{\boldsymbol{n}}}$ in terms of $\boldsymbol{\omega}^{(\mathrm{acc})}$ and $\boldsymbol{e}_{\mathrm{s}}$.

The difference between the approximation (16) and (17) and the full transformation of the acceleration system is, thus, that the rotation rate in the direction of the acceleration is not accounted for in (16) and (17).

2.4. Circular flows

Flows that can be described in a cylindrical $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{z}})$ co-ordinate system and that are homogeneous in the $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{z}}$ directions are truly 3D if the $\hat{\boldsymbol{z}}$ component of the mean velocity is not constant in the $\hat{\boldsymbol{r}}$ direction. Fully developed pipe flow rotating around the symmetry axis and a free vortex with a velocity deficit are examples of such flows. The advection of the anisotropy represented in the $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{z}})$ system vanishes exactly and the corresponding co-ordinate system rotation rate vector is $\boldsymbol{\omega}^{(\mathrm{r})} = (V(r)/r)\hat{\boldsymbol{z}}$, where $V(r)$ is the (absolute) angular velocity. The $\boldsymbol{\omega}^{(\mathrm{r})}$ vector is, thus, only constant if the flow is a rigid body rotation. In general $\boldsymbol{\omega}^{(\mathrm{r})}$ is a function of r.

This class of flows may be used for testing if the fully 3D forms of the proposed corrections are consistent with the exact one. It is shown in Appendix A that both the proposed correction based on the strain-rate tensor in (12) as well as the one based on the acceleration in (16) are identical to the exact one. Spalart and Shur (1997) also proposed a 3D correction which is different from that proposed here and that fails in reproducing the exact $\boldsymbol{\omega}^{(\mathrm{r})}$, see Appendix A.

3. Model calibration

Different sets of $A_{0\text{–}4}$ coefficients in Eq. (1) will be tested for some generic test cases where rotational effects are significant. The EARSM derived from the linearized SSG (Speziale et al., 1991) denoted 'L-SSG' has the coefficients shown in Table 1. That model, without curvature corrections, was proposed by Girimaji (1996). The Wallin and Johansson (2000) EARSM, Wallin and Johansson (WJ), was developed without any corrections related to the l.h.s. of Eq. (1) and, thus, the resulting model was independent of the A_0 coefficient. However, the value for A_0 resulting from the original model is shown in the table. For some of the comparisons, the curvature correction for the WJ model will be switched off to let the flow be represented in an inertial system. That corresponds to $A_0 \to \infty$ and is denoted as 'iWJ' in the table.

Introducing the curvature correction for the original choice of the A_0 coefficient in the WJ model leads to a model that predicts rotational effects poorly as will be seen later in this paper and also observed by Wallin and Girimaji (2000) for the vortex flow. Wallin and Girimaji found that the WJ model behaviour was improved by increasing A_0 to a value closer to that of the L-SSG.

Table 1
The values of the A coefficients for different quasi-linear pressure–strain models

	A_0	A_1	A_2	A_3	A_4
L-SSG	−0.80	1.22	0.47	0.88	2.37
WJ	−0.44	1.20	0	1.80	2.25
iWJ	∞	1.20	0	1.80	2.25
CC-WJ	−0.72	1.20	0	1.80	2.25

Here, we will do a more thorough analysis of the effect of the A_0 coefficient. The resulting final calibration of the model is denoted CC-WJ in the table.

In calibrating the A_0 coefficient, the long time asymptotic behaviour in rotating homogeneous shear flow is considered. Fig. 2 shows the effective C_μ in the asymptotic limit for different rotation numbers, $Ro \equiv \omega_z^{(\mathrm{r})}/(\mathrm{d}U/\mathrm{d}y)$, and different models. The effective C_μ is defined from

$$\frac{\mathscr{P}}{\varepsilon} = C_\mu^{(\mathrm{eff})}\left(\frac{K}{\varepsilon}\frac{\partial U}{\partial y}\right)^2. \tag{18}$$

Depending on the rotation number the turbulent kinetic energy grows exponentially with constant $\mathscr{P}/\varepsilon = (C_{\varepsilon 2} - 1)/(C_{\varepsilon 1} - 1)$ (for non-zero effective C_μ) or follows a power-law solution where $\varepsilon/(U_y K) \to 0$ (Speziale and Mac Giolla Mhuiris, 1989).

The bifurcation points between the two solution branches, Ro^- and Ro^+, correspond to the points where $C_\mu^{(\mathrm{eff})}$ becomes zero or where the flow is close to neutral stability. There is, however, a weak algebraic growth associated with the power-law solution very close to the bifurcation points (see Durbin and Pettersson-Reif, 1999).

Neutral stability occurs near $Ro = 0.5$ and is also likely associated with the linear velocity profile in the core of a rotating channel (local $Ro \approx 0.5$) according to Pettersson-Reif et al. (1999) (see also Oberlack, 2001). Thus, one might calibrate the A_0 coefficient such that the required bifurcation point Ro^+ is obtained. Such a relation reads

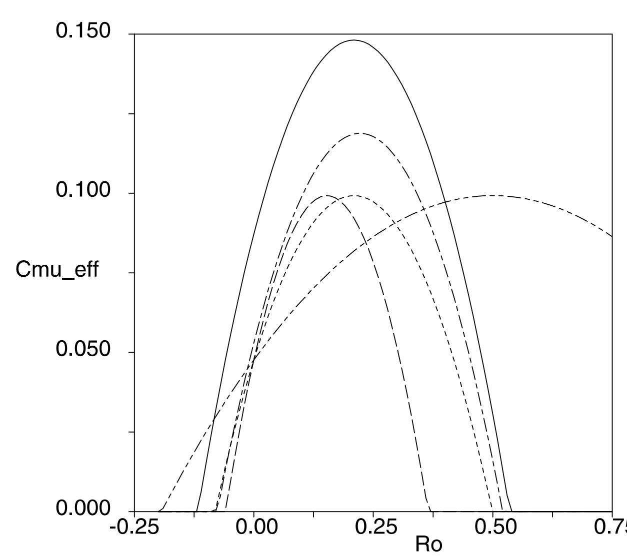

Fig. 2. Computed effective C_μ for rotating homogeneous shear flow in equilibrium (asymptotic solution). The models are curvature corrected WJ (– – –), CC-WJ (- - -) and L-SSG (– - – - –) EARSMs compared to non-corrected iWJ EARSM (– - - - - - –). Also CC-WJ for $\mathscr{P} = \varepsilon$ is shown (—).

$$\frac{2Ro^+}{A_0} = 2Ro^+ - 1 - \sqrt{\frac{1}{2}A_1A_3\frac{C_{\varepsilon 1}-1}{C_{\varepsilon 2}-1} + \frac{1}{2}A_1A_4 + \frac{1}{3}A_2^2} \qquad (19)$$

which gives $A_0 = -0.72$ for $Ro^+ = 0.5$ with the WJ values in Table 1 for the A_{1-4} coefficients and $C_{\varepsilon 1} = 1.44$ and $C_{\varepsilon 2} = 1.83$. That value is used for the CC-WJ model in Table 1 and the resulting effective C_μ is given in Fig. 2.

The DNS data of the rotating channel by Alvelius and Johansson (1999) shows that there is an approximate equilibrium ($\mathscr{P} \approx \varepsilon$) maintained by turbulent transport effects. One can also observe that the effective C_μ is small, but finite and non-constant through the core region. This is consistent with the CC-WJ curve in Fig. 2 for $\mathscr{P} = \varepsilon$. At $Ro = 1/2$ the model predicts a small, but finite effective C_μ and with a large negative derivative for increasing Ro. The slope is important, since this implies that if the velocity gradient in the core is given a small positive perturbation, then Ro is slightly decreased giving an increased effective C_μ. This leads to an increased effective eddy viscosity that drives the velocity gradient back to the equilibrium state, because of the balance in the momentum equation. Thus, balance is obtained close to the Ro^+ point, but within the region of exponential asymptotic growth.

4. Generic test cases

4.1. Rotating homogeneous shear flow

Rotating homogeneous shear flow may be used as an illustration of the effect of including the streamline curvature correction. The flow is rotating with the rate $\omega_z^{(r)}$ in the $\hat{z}$ direction. In this specific case it is obvious to transform the anisotropy tensor to the rotating coordinate system. Exactly the same effect is obtained by applying the corrections proposed in this paper, and the weak-equilibrium assumption is then exactly fulfilled.

Four different cases were computed for rotation rates $Ro \equiv \omega_z^{(r)}/(\mathrm{d}U/\mathrm{d}y)$ of 0, 1/4, 1/2 and −1/2 (see Fig. 3). It is obvious that the eddy-viscosity model cannot distinguish between the different rotation rates.

These cases were computed with the different curvature corrected algebraic Reynolds stress models given in Table 1. The model based on the L-SSG gives reason-

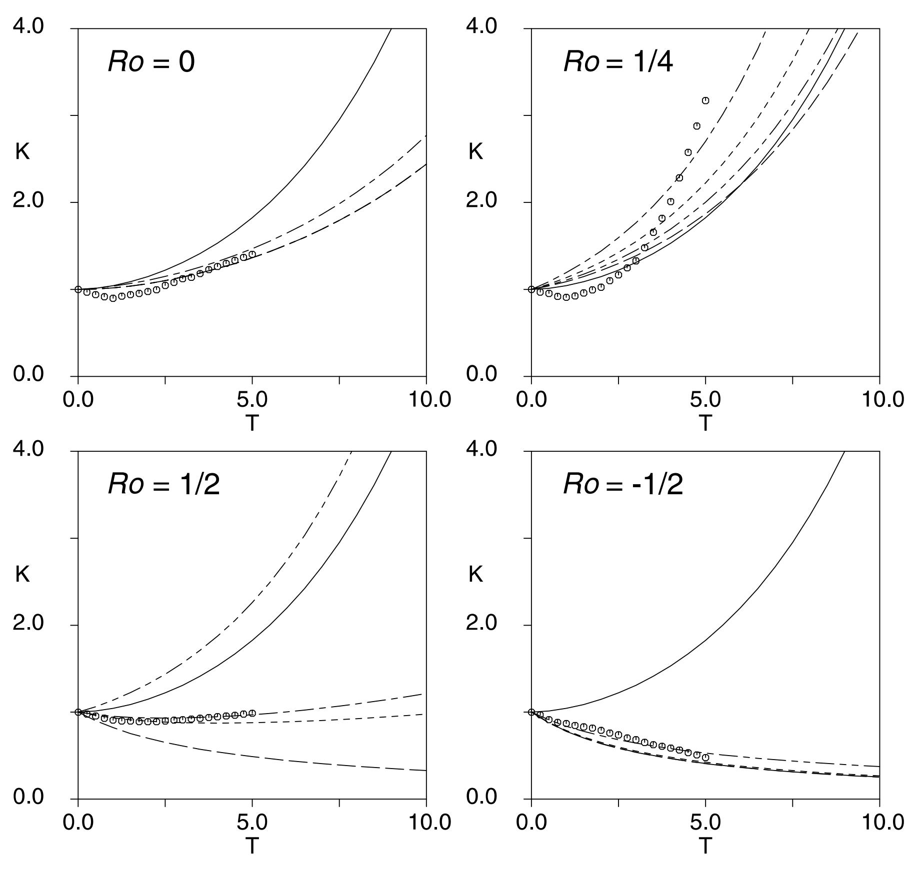

Fig. 3. Computed evolution of the turbulent kinetic energy K in rotating homogeneous shear flow compared to large eddy simulation (○) (Bardina et al., 1983). Curvature corrected WJ (– – –), CC-WJ (- - -) and L-SSG (– - – - –) EARSMs compared to non-corrected iWJ EARSM (– - - - - - –) and eddy-viscosity model (—). $\tau \mathrm{d}U/\mathrm{d}y = 3.4$ initially.

able growth rates for most rotation numbers, though underestimating the most energetic case ($Ro = 1/4$). The WJ model underestimates both cases with positive rotation. However, by increasing the A_0 coefficient the CC-WJ model gives predictions close to the L-SSG model.

Switching off the curvature correction, as in the iWJ model, degenerates the predicted growth rate for the $Ro = 1/4$ case, and for the $Ro = 1/2$ case the growth rate is severely overpredicted. From this, it is clear that the streamline curvature correction is important. Since the A_0 coefficient is arbitrary without the curvature correction, the CC-WJ and WJ models are identical without correction or curvature effects.

4.2. Fully developed rotating channel

Fully developed rotating channel is considered. The channel co-ordinate system is $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ which is rotating with the rate $\omega_z^{(\mathrm{r})}$ in the $\hat{\boldsymbol{z}}$ direction. Also in this case it is obvious to transform a_{ij} to the rotational frame, and both the exact transformation and the proposed approximation exactly fulfill the weak-equilibrium assumption concerning the advection of the anisotropy tensor.

Direct numerical simulations of a fully developed rotating channel at different Reynolds and rotational numbers were made by Alvelius and Johansson (1999). The two most rapidly rotating cases for $Re_\tau \equiv u_\tau\delta/\nu = 180$ are computed here. δ is the half channel width and the average wall friction velocity is defined as $2u_\tau^2 = (u_\tau^{\mathrm{s}})^2 + (u_\tau^{\mathrm{u}})^2$ where u_τ^{s} and u_τ^{u} are the stable and unstable side friction velocities, respectively. The rotation number $Ro \equiv 2\omega_z^{(\mathrm{r})}\delta/U_m$ and the bulk Reynolds number $Re_m \equiv U_m\delta/\nu$ are given in Table 2, where $\omega_z^{(\mathrm{r})}$ is the rotation rate of the system and U_m is the bulk velocity.

Table 2
Rotating channel flow. DNS specification (Alvelius and Johansson, 1999) and computational results using the curvature corrected CC-WJ EARSM

	DNS	DNS	CC-WJ	CC-WJ
Ro	0.43	0.77	0.43	0.77
Re_τ^{s}	129.7	133.0	129.8	138.4
Re_τ^{u}	218.3	217.2	218.8	213.4
Re_m	3094	3446	3257	3804

The Wallin and Johansson (2000) EARSM together with the Wilcox (1988) K–ω model is computed with the proposed curvature correction. The curvature corrected CC-WJ EARSM agrees well with DNS data, though a somewhat overpredicted Re_m (for the prescribed Re_τ) is seen in the U^+ plots in Fig. 4 and in Table 2. Using the curvature corrected original WJ EARSM the effect of rotation is slightly overestimated while the non-corrected iWJ EARSM underpredicts the rotation effects. Thus, the effect of curvature correction and the

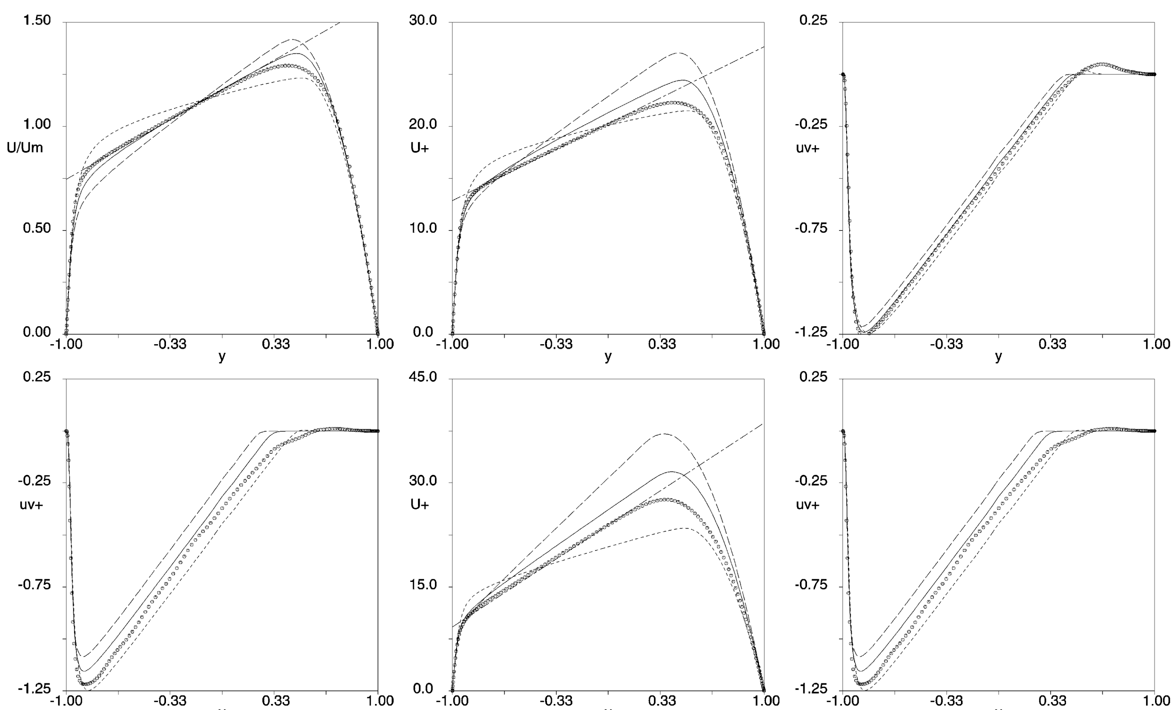

Fig. 4. Computed rotating channel flow for $Ro = 0.43$ (top) and $Ro = 0.77$ (bottom) compared to DNS of Alvelius and Johansson (1999). Curvature corrected WJ (– – –) and CC-WJ (—) EARSM compared to non-corrected iWJ EARSM (- - -). $U' = 2\omega_z^{(\mathrm{r})}$ (– - –) is also shown.

choice of A_0 are important also for this case. In Table 2 it is seen that the skin friction differences between the stable and unstable sides are well captured by the curvature corrected CC-WJ model.

The shear stress plots in Fig. 4 show that all models capture the laminarization on the stable side reasonably well. However, the DNS data show a small level of positive shear stress for the $Ro = 0.43$ case where all models give almost zero $\overline{uv}$.

5. Concluding remarks

The local mean velocity gradient is not sufficient in determining the curvature or rotational effects and an independent additional measure is needed. Two basically different options for determining that measure are suggested by considering the rate of change of either the strain-rate tensor or the acceleration vector. In both methods, the second order derivative of the velocity field is needed. In situations where the strain rate or the acceleration vanishes the corrections might become almost singular. That situation is less severe for the strain-rate based method, since the turbulence production and the influence of the turbulence is less critical in vanishing mean flow strain rate. Hellsten (submitted for publication) has found that the acceleration based method leads to problems in some situations of mild curvature where the direction of the acceleration vector may vary rapidly. This was demonstrated in a U-bend flow, which showed an almost singular behaviour. For the same case, the strain-rate based method behaves much better.

The strain-rate tensor and the acceleration vector, as well as their material derivatives, fulfill Galilean invariance, that is independence of solid-body motion of the frame of reference, and, thus, also the proposed corrections are invariant. However, any incompressible flow field should also be independent of a superimposed solid-body constant acceleration, according to Spalart and Speziale (1999), except for a modified pressure field. The acceleration based modification must thus be used with caution in accelerated frames of reference. Extensions of EARSMs for including approximations of the usually neglected transport terms from the l.h.s. of Eq. (1) could never be expected to be completely general, but could anyway be motivated by improved model performance in a reasonably wide class of flows.

The EARSM proposed by Wallin and Johansson (2000) was originally without any corrections originating from the l.h.s. of Eq. (1) and, thus, the choice of the A_0 coefficient was arbitrary. Introducing the proposed curvature correction resulted in a rather poor behaviour in rotating flows in contrast to the L-SSG. By calibrating the A_0 coefficient for consistency with neutral stability in irrotational flows, a value closer to that of the L-SSG was obtained. The behaviour of the resulting model, CC-WJ, became quite similar to the L-SSG without influencing the behaviour of the original WJ model in flows without curvature effects. The rather limited complexity of the fully 3D WJ EARSM, compared to the L-SSG, is, thus, retained also with the modified A_0 coefficient and with the curvature correction. In the WJ EARSM, $A_2 = 0$ ($C_3 = 2$) which means that the last term in (1) vanishes leading to a reduction in the algebraic complexity for the resulting EARSM, especially in 3D flows (see Wallin and Johansson, 2000).

Table 3
The values of the C coefficients for different quasi-linear pressure–strain models

	C_1^0	C_1^1	C_2	C_3	C_4
L-SSG	3.4	1.8	0.36	1.25	0.40
WJ	3.6	0	0.8	2	1.11
CC-WJ	4.6	1.24	0.47	2	0.56

It is, however, important to highlight that the corresponding pressure–strain rate model, given by Eq. (2), becomes completely different by a modification of the A_0 coefficient. The coefficients are given in Table 3. The CC-WJ model introduces e.g. a non-linear part through the non-zero C_1^1 and also, the value of the C_2 coefficient is not consistent with the rapid distortion theory, which gives $C_2 = 4/5$. Also the L-SSG model has a departure from the theoretical $C_2 = 4/5$ even though the full non-linear SSG fulfills the rapid distortion theory. It is, thus, reasonable that a lower value of the C_2 coefficient is a better compromise over a wider parameter range than the more extreme state of rapid distortion and that a non-linear model is needed for covering also the rapid distortion limit.

The test cases considered in this paper were chosen primarily for validating the calibration of the model coefficients. The cases also clearly demonstrate the importance of applying curvature correction even if the curvature is explicitly known for these particular cases. The novel feature in the proposed approximation methods of the local curvature in general flows are mainly the explicit 3D forms. Hopefully, this paper will encourage the use of the proposed corrections in more complex 3D flows.

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Appendix A. Circular flow with an axial component

Circular flows with an axial component is described in a cylindrical $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{z}})$ system. The velocity is given by

$$\boldsymbol{U} = V(r)\hat{\boldsymbol{\theta}} + W(r)\hat{\boldsymbol{z}}. \tag{A.1}$$

The exact transformation may be determined to

$$\boldsymbol{\omega}^{(\mathrm{r})} = \frac{V}{r}\hat{\boldsymbol{z}}. \tag{A.2}$$

The acceleration and rate of change of that are

$$\dot{\boldsymbol{U}} = -\frac{V^2}{r}\hat{\boldsymbol{r}}, \quad \ddot{\boldsymbol{U}} = -\frac{V^3}{r^2}\hat{\boldsymbol{\theta}}. \tag{A.3}$$

The approximation of the system rotation rate based on the acceleration system, $\boldsymbol{\omega}^{(\mathrm{approx})}$ in (16) becomes

$$\boldsymbol{\omega}^{(\mathrm{approx})} = \frac{V}{r}\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}} = \frac{V}{r}\hat{\boldsymbol{z}} \tag{A.4}$$

which is identical to the exact transformation.

The strain-rate tensor is then given by

$$\boldsymbol{S} = \frac{1}{2}\begin{pmatrix} 0 & V' - \frac{V}{r} & W' \\ V' - \frac{V}{r} & 0 & 0 \\ W' & 0 & 0 \end{pmatrix} \tag{A.5}$$

and the rate of change of that is

$$\dot{\boldsymbol{S}} = \frac{V}{r}\begin{pmatrix} \frac{V}{r} - V' & 0 & 0 \\ 0 & V' - \frac{V}{r} & \frac{1}{2}W' \\ 0 & \frac{1}{2}W' & 0 \end{pmatrix}. \tag{A.6}$$

The approximation based on $\dot{\boldsymbol{S}}$, $\boldsymbol{\omega}^{(S)}$ in (12), can be determined by use of the $\boldsymbol{S}$ and $\dot{\boldsymbol{S}}$ relations

$$\boldsymbol{\omega}^{(S)} = \frac{V}{r}\hat{\boldsymbol{z}} \tag{A.7}$$

which also is identical to the exact transformation.

The Spalart–Shur correction e in 3D can be written as

$$e = \frac{S_{pl}\dot{S}_{lq}\epsilon_{pqi}}{2II_S}(2\omega_i), \tag{A.8}$$

where ω_i is the vorticity. The quantity e is the inner product of the rotation rate of $\boldsymbol{S}$ following the streamline and $\boldsymbol{S}$ rotated by the rate of the vorticity vector. That can be written as

$$e = \omega_i^{(S\text{–}S)}(2\omega_i) \tag{A.9}$$

and by identification, the approximation of the system rotation rate based on Spalart–Shur is given by

$$\omega_i^{(S\text{–}S)} = \frac{S_{pl}\dot{S}_{lq}\epsilon_{pqi}}{2II_S}. \tag{A.10}$$

By including the $\boldsymbol{S}$ and $\dot{\boldsymbol{S}}$ relations, one obtaines

$$\begin{aligned} \omega_r^{(S\text{–}S)} &= 0, \\ \omega_\theta^{(S\text{–}S)} &= \frac{\frac{3}{4}W'\frac{V}{r}\left(\frac{V}{r} - V'\right)}{\left(\frac{V}{r} - V'\right)^2 + W'2}, \\ \omega_z^{(S\text{–}S)} &= \frac{V}{r}\left[1 - \frac{\frac{3}{4}W'2}{\left(\frac{V}{r} - V'\right)^2 + W'2}\right]. \end{aligned} \tag{A.11}$$

Hence, the Spalart–Shur correction reduces to the exact transformation only in cases where the axial velocity is constant ($W' = 0$).

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